Competition rules:

(a) The competition has six questions, each counting for ten points. You have 180 minutes to solve as many as you can correctly.

(b) An answer without proof or reasoning is not complete and will receive little to no points.

(c) On each page you should clearly write your name and the number of the question. On at least one page you should write your year and what you study.

(d) Books, calculators, phones and other utensils are not permitted. Only pen and paper can be used.

(e) For further questions or disputes only the jury is qualified.

(f) By participating to PUMA 2023 you agree to these rules.

(g) Have fun and succes!

The results will become available in the evening of Thursday March 16th on the Wina-site

Questions

1 Friend network

200 mathematics students are using Facebook. Each of them is friends with at least n other students.

(a) *(5 points)* How large does n minimally have to be such that every two students are either friends or have a common friend?

(b) *(2.5 points)* Prove that, at this minimal n, students who are not friends have at least two common friends.

(c) *(2.5 points)* At this minimal n, do students who are friends always have a common friend?

2 Trigonometry

If $\sin^3\theta + \cos^3\theta = \frac{11}{16}$, what does $\sin\theta + \cos\theta$ equal?

3 Tricky triangles

Show that for $n \geqslant 6$, we can divide an equilateral triangle in n non-overlapping equilateral triangles (which may have a different size).

4 Chess board

(a) *(4 points)* Consider an 8×8 chess board, of which the lower left 3×3-corner is filled with frogs, where each frog takes up one square. A frog moves by jumping over another frog standing on an adjacent square. Can the frogs move in such a way that they end up in the upper right 3×3-square?

(b) *(6 points)* Show that a knight cannot visit all squares of a $4 \times n$ chess board exactly once, and then end at the square it started at. (A knight moves according to the following rule: starting from one square, it can move two squares in either direction and then 1 square perpendicular to the direction it just moved in. The only squares it "visits" are the first one and the last one.)

5 Pseudo-invertible matrix

Let $A \in \mathbb{R}^{m\times n}$ be a matrix. Given a vector $X \in \mathbb{R}^{m\times 1}$, show that there exists a vector $Y \in \mathbb{R}^{n\times 1}$ such that $A^T AY = A^T X$.

6 Interesting integral

Let a, b be positive real numbers. Calculate

$$\int_a^b \frac{e^{\frac{x}{a}} - e^{\frac{b}{x}}}{x} \mathrm{d}x$$

Solutions

1 Friend network

(a) The minimal such n is 100. We first show by contradiction that every two students or either friends or have a commob friend. Let A and B be two students who are not friends and do not have any common friends. Then each of them has at least 100 friends, which all have to be distinct. But then there would have to be at least 202 students: student A, student B and their 200 friends.

We can show that $n = 99$ does not work by means of a counterexample. Choose 198 students who are all friends with each other. The remaining two students can then each be friends with exactly 99 of the other students without any overlap, such that these two students aren't friends and don't have a common friend.

(b) If these two students both have 100 friends of which only one is common, there would be at least $2 + 200 - 1 = 201$ students in total.

(c) Not necessarily: choose 198 students who are all friends. Let the remaining two students, A and B, be friends; divide the group of 198 students in two disjoint groups of 99 students. Let A be friends with all students from one group and B with all students from the other group. Then A and B are friends but do not have a common friend.

2 Trigonometry

$$\begin{aligned}\frac{11}{16} &= \sin^3\theta + \cos^3\theta \\ &= (\sin\theta + \cos\theta)(\sin^2\theta - \sin\theta\cos\theta + \cos^2\theta) \\ &= (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta)\end{aligned}$$

Let $x = \sin\theta + \cos\theta$. Note that

$$x^2 = (\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = 1 + 2\sin\theta\cos\theta$$

Hence $\sin\theta\cos\theta = \frac{x^2-1}{2}$. By the first equality it follows that

$$\frac{11}{16} = x\left(1 - \frac{x^2-1}{2}\right)$$

We conclude that

$$8x^3 - 24x + 11 = 0$$

It is straightforward to verify that $x = \frac{1}{2}$ and $x = \frac{-1\pm3\sqrt{5}}{4}$ are the roots of this polynomial equation. Note that as $x^2 = 1 + \sin 2\theta \leq 2$, only $x = \frac{1}{2}$ works.

3 Tricky triangles

Note that we can easily divide the triangle into four equal parts by connecting the midpoints of all sides. This means that if a division in k parts is possible, so too is a division in $k + 3$ parts: indeed, we can divide one of the triangles in four to get three more triangles. Because of this, it suffices to find divisions in 6, 7 or 8 triangles. These are given in Figure 1 below:

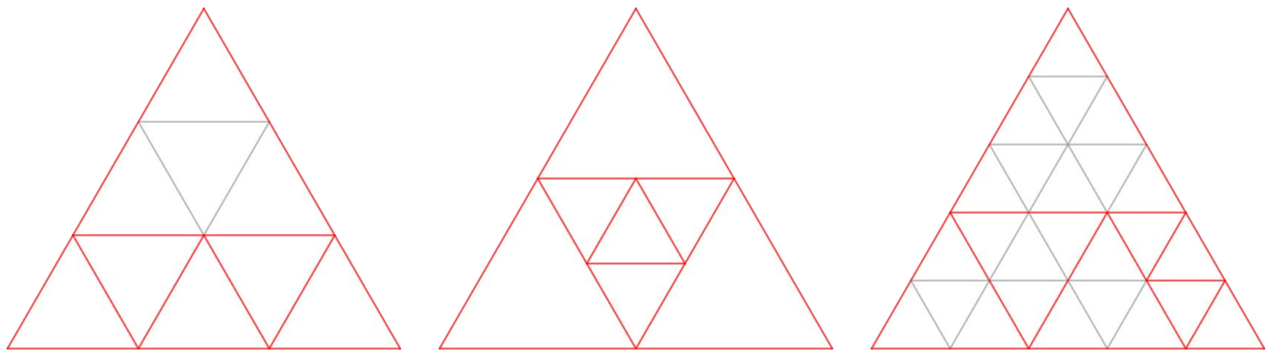

Figure 1: Division in 6, 7 or 8 triangles.

4 Chess board

(a) Colour the rows alternatingly blue and red. Note that a frog always jumps to a square of the same colour. Because the number of red and blue squares differs in the proposed start and end position, it is in fact impossible.

(b) Colour the upper and lower row blue and the middle two rows red. Note that a knight on a blue square can only jump to a red square. There is an equal amount of blue and red squares, so if it makes a closed walk in which no square (but the starting square) gets visited once, it must alternate between blue and red squares.

Now colour the chess board black and white as usual. A knight must always alternate between black squares and white squares. This means that in the red zone, the knight can only visit either the white or the black squares, meaning the closed walk is indeed impossible.

5 Pseudo-invertible matrix

Note that it suffices to show that the column space (vector space generated by the columns) of A^T equals that of A^TA. Clearly any element of the column space of A^TA is an element of the column space of A^T, so if we show that their dimensions are equal, we are done.

This amounts to showing $\mathrm{rank}(A^T) = \mathrm{rank}(A^TA)$. Now the rank of A^T equals that of A, which equals n minus the dimension of its null space. Similarly the rank of A^TA is n minus the dimension of its null space. Hence it suffices to show that the null spaces of A and A^TA are equal. Clearly the null space of A is contained in that of A^TA; conversely, if $A^TAv = 0$, we get $v^TA^TAv = (Av)^TAv = 0$, whence $Av = 0$. This finishes the proof.

6 Interesting integral

We have

$$\int_a^b \frac{e^{\frac{x}{a}} - e^{\frac{b}{x}}}{x}\mathrm{d}x = \int_a^{\sqrt{ab}} \frac{e^{\frac{x}{a}} - e^{\frac{b}{x}}}{x}\mathrm{d}x + \int_{\sqrt{ab}}^b \frac{e^{\frac{x}{a}} - e^{\frac{b}{x}}}{x}\mathrm{d}x,$$

Substituting $y = \frac{ab}{x}$ in the second integral:

$$\begin{aligned} &= \int_a^{\sqrt{ab}} \frac{e^{\frac{x}{a}} - e^{\frac{b}{x}}}{x}\mathrm{d}x - \int_{\sqrt{ab}}^a \frac{e^{\frac{b}{y}} - e^{\frac{y}{a}}}{y}\mathrm{d}y \\ &= \int_a^{\sqrt{ab}} \frac{e^{\frac{x}{a}} - e^{\frac{b}{x}}}{x}\mathrm{d}x - \int_a^{\sqrt{ab}} \frac{e^{\frac{x}{a}} - e^{\frac{b}{x}}}{x}\mathrm{d}x = 0. \end{aligned}$$